Estimation Theory for a Single Server Queue With Random Arrivals and Complete Balking

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ABSTRACT

This paper describes parameter estimation for the queueing system investigated by Rubin and Robson (1989a), which consists of Poisson input of customers, some of whom are lost to balking; and a single server working a shift of length L, providing a service whose duration can vary from customer to customer, and, if necessary, working overtime to complete the final service of the shift. Our focus is estimation of parameters of the process from observation of a single shift. Point and interval estimators of the unknown number of arrivals (n) or the unknown rate (λ) of the Poisson arrival process can be derived from the conditional and unconditional distributions, respectively, of total server idle time (T) or the number of services (X). Confidence limits are derived using the cumulative distribution functions of T or X, while maximum likelihood estimators are derived using the density functions of T or X. For the case of equal service time, point estimators of n, based on T or X, can be constructed to be unbiased over the restricted range of $n < L/w$. Point and interval estimators of n can be derived using the conditional cdf of the number of balkers for the case of equal service time, as well. The conditional distributions also can be used to estimate shift length when n and X or n and T are observed. Both maximum likelihood estimates and confidence limits for L are derived.

KEY WORDS: Queue; Interval estimation; Maximum likelihood estimation.

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1. INTRODUCTION

This paper describes estimation of the parameters of a queueing process, based on observation of a single shift, for several situations in which incomplete data are collected. The queueing system to be considered consists of Poisson input of customers, some of whom are lost to balking, and a single server working a shift of length L and providing a service whose duration, w_i, can vary from customer to customer (Rubin and Robson, 1989a). If a service is in progress at the end of a shift, the server works overtime to complete the final service.

Investigation of this queueing system was motivated by the behavior of fishermen encountered in the New York Great Lakes Creel Survey (Robson and Jones, 1989). In this study, an interviewer, working a shift of length L, is stationed at a marina or boat launch and asks fishermen returning with their catch a fixed set of questions, requiring constant service time, w. Balking arises since a fisherman will leave immediately if the interviewer is occupied. Although the model described above will allow for variable service time, the length of the interview was virtually constant for all fishermen. If a service is in progress at the end of a shift, the server works overtime to complete the service. For the Creel Survey, the goal is to estimate the unknown number (n) of fishermen returning to the marina during the shift, based on the known number of interviews. Once the number of arrivals is estimated, the number of fish caught can be estimated for the balkers, thereby providing the estimates of the total catch for each fish species. However, our discussion of estimation is not limited to what is appropriate for the Creel Survey.

The service times are assumed to be independent, identically distributed, strictly positive random variables. However, for purposes of estimation, it is useful to realize that the service time distribution also can be viewed in terms of sequences of w_i's that are known through the final service of the shift. The latter perspective allows one to estimate n or arrival rate λ when only the sequence of service times through the final service has been recorded for a shift. Estimation for the special case of constant service time, w, for all customers will be discussed where this simpler case affords results unavailable in general.

Point and interval estimators of the unknown number of arrivals (n) or the unknown rate (λ) of the Poisson arrival process are derived from the conditional and unconditional distributions, respectively, of total server idle time (T), the joint distribution of the number of services and overtime (X, Y) or the number of services (X). Estimation of n based on the conditional cdf of the number of balkers is discussed for the case of equal service times. In addition, the conditional distributions are used to estimate shift length (L) when n and X or n and T have been observed. Estimation of common service time for this queueing process is discussed in Rubin (1987) and in Rubin and Robson (1989b).

Samaan and Tracy (1979) derive estimators of the customer arrival rate for a single server queue with loss (balking) when the customer arrivals form a Poisson process and the service times are uniform on the interval (0, 1). In their case, the interdeparture times of customers are known and become the basis for estimation of λ. Assefi (1979) and Basawa and Prakasa Rao (1980) provide good overviews of estimation and statistical inference for stochastic processes.

For notational convenience, variables, moments, probabilities and distributions, which are conditional on the realized number (n) of arrivals, will be denoted with a lowercase subscript (n), while their unconditional counterparts bear an uppercase subscript (N). We use the term probability density function (pdf) loosely, applying it to mixed distributions as well as to continuous distributions. The relevant distributional results for this queueing system, which are derived in Rubin and Robson (1989a), will be stated, when necessary, to facilitate the derivation of estimators.

2. ESTIMATION WHEN NUMBER OF SERVICES (X) AND OVERTIME (Y) OR TOTAL IDLE TIME (T) ARE OBSERVED

When the number of services and the amount of overtime required to complete the last service are both observed, the joint density functions of X and Y can be used as the basis for constructing estimators: the joint distribution, conditional on the number of arrivals, yields an estimator for n, while the unconditional joint distribution yields an estimator for λ. When the sequence of service times is known through the x^{th} service, then $T=L-W_X+y$ uniquely specifies X (Rubin and Robson, 1989a). In those cases, it will be found preferable to use the conditional and unconditional density functions of total server idle time (T) as a statistical basis for estimation of n and λ, respectively.

Estimation based on total server idle time (or its complement, the cumulative service time for the shift) is not well motivated by the Creel Survey. Instead, consider the following industrial setting in which the amount of time a particular machine is free (or busy) during a shift is recorded. The customers are the workers, who periodically require use of the machine, and balking occurs if the plant policy is that workers do not wait for the machine if it is in use.

2.1 Maximum Likelihood Estimator of n

The maximum likelihood estimator (MLE) of the number of arrivals can be constructed either from the conditional density function of T_n, given by

$$f_{T_n}(t;\ n, L) = \begin{cases} \binom{n}{x} t^x (L-t)^{n-x} / L^n & \text{for } t = L-W_x \\ x \binom{n}{x} t^{x-1}(L-t)^{n-x} / L^n & \text{for } \max(0, L-W_x) < t < L-W_{x-1} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\max(0, L-W_n) \le t < L$, $0 < w_x$ or from the conditional joint density function of X_n and Y_n, given by

$$f_{X_n, Y_n}(x, y; n, L) = \begin{cases} \binom{n}{x}(L-W_x)^x W_x^{n-x}/L^n & \text{for } W_x < L,\ y = 0 \\ x\binom{n}{x}(L-W_x+y)^{x-1}(W_x-y)^{n-x}/L^n & \text{for } \max(0, W_x - L) < y < w_x \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Notice that one must know the sequence of w_i's through x to use the former, while one need know only the cumulative service time, W_x, to use the latter. In either case, setting the difference between the likelihood functions at n and n-1 equal to zero yields:

$$\hat{n}_{ml}(T) \equiv x\,L/T = X\,L/(L-W_x+Y).$$

The ratio of the likelihoods,

$$f_n(t_x)/f_{n-1}(t_x) = (n/(n-x))((L-t_x)/L) = (n/(n-x))((W_x - y)/L),$$

where $0 < L - t_X = W_X - y < L$, is a strictly decreasing function of n, which passes through unity at a point $n = \hat{n}_{ml}(T)$ that is relatively close to x for small W_X. The ratio of adjacent ratios of the likelihoods is less than unity:

$$(f_{n+1}(t_x)/f_n(t_x))/(f_n(t_x)/f_{n-1}(t_x)) = 1 - x/(n(n-x+1)) < 1$$

for $0 \le x \le n$, implying that the likelihood function is unimodal with its maximum near $\hat{n}_{ml}(T) = X\,L/(L-W_x+Y)$. Numerical results indicate that the variance estimator:

$$\widehat{Var}(\hat{n}_{ml}(T) \mid N = n) = \hat{n}_{ml}(T)(\hat{n}_{ml}(T) - X)/X = XL(L-T)/T^2$$

(based on second differencing) is improved by replacing $\hat{n}_{ml}$ with $\hat{n}_{ml}$+0.5.

2.2 Construction of a Restricted Unbiased Estimator of n

A point estimator, which is unbiased over the restricted range of n less than M: $W_{M-1} \leq L < W_M$, can be constructed by noting that for such values of n neither the range of X_n nor the range of Y_n is dependent on L, and hence, the first derivative of the conditional log likelihood with respect to L must have zero expectation for all such values of n. This calculation can be implemented on the joint conditional likelihood of (X_n,Y_n), given in (2), or on the conditional pdf of T_n, given in (1). Expressed in terms of the busy time in [0, L], $S = W_x - Y$, and the number of services during the busy period,

$$x(S) = \begin{cases} x & \text{if } S = W_x \\ x-1 & \text{if } W_{x-1} < S < W_x \end{cases},$$

this gives:

$$E\left(d \ln f_{X_n, Y_n}(x, y) / dL\right) \equiv 0 = E\left(\{x(S)/(L-S)\} - n/L\right)$$

or

$$E\left(L\, x(S)/(L-S)\right) = n\,,$$

for nonnegative integer values of n less than M: $W_{M-1} \leq L < W_M$. In terms of previous notation,

$$(L-S)/L = T/L\,,$$

which represents the fraction of the shift during which the server is idle. Thus, we have proved:

Theorem 1.

$$\tilde{n}(S) = \tilde{n}(X, Y) = L\,x(S)/(L-S)$$

is unbiased for $n < M$: $W_{M-1} \le L < W_M$, where

$$x(S) = \begin{cases} x & \text{if } S = W_x \\ x-1 & \text{if } W_{x-1} < S < W_x \end{cases} .$$

The corresponding estimator based on T, $\tilde{n}(T) = xL / T$, is also unbiased for $n < M$. Both estimators exceed M essentially whenever the idle time constitutes less than 50 percent of the shift. This feature renders $\tilde{n}(T)$ and $\tilde{n}(S)$ unacceptable for estimation of n.

Note, however, that if n is also an observed datum, then

$$\hat{L}_{ml}(S) = S/(1 - x(S)/n),$$

while if n is not observed and L is unknown then identifiability is lost. Estimation of L is discussed further in Section 5.

2.3 Interval Estimation of n

The cdf of idle time, conditional on n arrivals, is an increasing function of n and can be used to construct confidence limits for n. A 1-α lower confidence limit for n can be constructed by solving for n in the equation:

$$\alpha = P(T_n \le t/L; n) = x^* \binom{n}{x^*} \int_0^{t/L} u^{x^*-1} (1-u)^{n-x^*} du , \qquad (3)$$

where $0 < t/L < 1$, $0 < w_x$ and

$$x^* = \begin{cases} x+1 & \text{for } L-t = W_x \\ x & \text{for } W_{x-1} < L-t < W_x . \end{cases}$$

Notice that the 2 possibilities for x^* arise because one must consider whether or not the upper limit of integration is equal to a mass point of the distribution.

We can transform the incomplete beta probability given in (3) to an F probability, so that lower and upper confidence limits for the number of arrivals (n) can be determined from the F-tables, using the appropriate confidence level. Applying the transformation

$$v = \left(n - x^* + 1\right) t \,/\, x^* \left(L - t\right)$$

to (3) gives

$$P\left(V_n \leq \left(n - x^* + 1\right) t \,/\, x^* \left(L - t\right)\right) = \alpha ,$$

where V_n has an F distribution with parameters $2x^*$ and $2(n-x^*+1)$. Notice that $t\,/(L-t)$ represents the estimated odds for service of any given one of the n randomly arriving customers, as does the unobserved ratio $(x^*/n)/\{1-(x^*-1)/n\}$. Thus, the 1-α lower confidence limit for n can be determined from the equation:

$$F_{2x^*,\, 2(n-x^*+1)}(\alpha) = \left(n - x^* + 1\right) t \,/\, x^* \left(L - t\right), \tag{4}$$

where $F_{a,\,b}$ is the critical value of the central F distribution with a and b degrees of freedom, for all t, $0 \leq t < L$.

The 1-α upper confidence limit for n can be obtained setting the upper tail probability of T_n equal to α. It can be shown that, for the conditional upper tail probability of T_n, $x^*=x+1$ for all t (Rubin, 1987). Therefore, the 1-α upper confidence limit for n is the solution to the equation

$$F_{2(x+1),\, 2(n-x)}\left(1 - \alpha\right) = (n - x)\, t \,/\, (x + 1)(L - t) . \tag{5}$$

It is more convenient to apply this definition of x^* to upper and lower confidence limits, alike. Thus, $1-2\alpha$ confidence limits for n can be constructed by holding the observed odds estimate, T/(L-T), fixed and adjusting the unobserved odds estimate, $\{(x+1)/n)\}/\{1-(x/n)\}$, to achieve odds ratios equal to upper and lower critical values of V_n.

Integer-valued approximate solutions to equations (4) and (5) can be determined using F-tables. Exact solutions, which are noninteger, can be computed using the F cdf and a root finding algoritm.

2.4 Maximum Likelihood Estimator of λ

The MLE of the rate parameter λ of the Poisson arrival process, $\hat{\lambda}_{ml}(T)$, can be constructed from the unconditional density function of T_N, given by

$$f_{T_N}(t;\lambda,L) = \begin{cases} (\lambda t)^x \exp(-\lambda t)/x! & \text{for } t = L-W_x \\ \lambda(\lambda t)^{x-1}\exp(-\lambda t)/(x-1)! & \text{for } \max(0, L-W_x) < t < L-W_x+w_x \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $0 \le t < L$, $0 < w_x$ and $\lambda > 0$,

or the unconditional joint density function of X_N and Y_N, given by

$$f_{X_N,Y_N}(x, y; \lambda, L) =$$

$$\begin{cases} \exp(-\lambda(L-W_x))\,(\lambda(L-W_x))^x/x! & \text{for } y=0,\ W_x < L \\ \lambda \exp(-\lambda(L-W_x+y))\,(\lambda(L-W_x+y))^{x-1}/(x-1)! & \text{for } \max(0, W_x-L) < y \le w_x \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Setting the first derivative with respect to λ of the unconditional likelihood function equal to zero yields:

$$\hat{\lambda}_{ml}(T) = x/T = X/(L-W_x+Y).$$

The second derivative with respect to λ of the likelihood function evaluated at $\hat{\lambda}_{ml}(T)$ is negative, indicating that $\hat{\lambda}_{ml}(T)$ is a maximum. Notice that the MLE's of n and λL are identical.

For the case of equal service time for all customers and $L\equiv 1$, $\hat{n}_{ml}(T)$ and $\hat{\lambda}_{ml}(T)$ both are bounded above by:

$$[1/w]/(1-w[1/w]) \equiv B(w) > [1/w](1+[1/w]).$$

Thus, the MLE's must underestimate a parameter that exceeds B(w). However, if X and Y are replaced by their asymptotic or exact expected values in $\hat{\lambda}_{ml}(T)$, an approximation to the expected value of $\hat{\lambda}_{ml}(T)$ is obtained:

$$E\left(\hat{\lambda}_{ml}(T)\right) \cong \left\{\lambda(1+E(Y_N))/(1+\lambda w)\right\}/\left\{1-\left(\lambda w\,(1+E(Y_N))/(1+\lambda w)\right)+E(Y_N)\right\} = \lambda\,;$$

the approximation improves in accuracy as B(w) approaches infinity. $\hat{\lambda}_{ml}(T)$ will underestimate parameter values exceeding B(w). However, B(w) is very large for small w.

The variance of $\hat{\lambda}_{ml}(T)$ is approximated by the inverse of the information, the expectation of the negative of the second derivative with respect to λ of the log likelihood function:

$$\operatorname{Var}\left(\hat{\lambda}_{ml}(T)\right) \cong \lambda^2/E(X_N). \tag{8}$$

An estimator of the variance of $\hat{\lambda}_{ml}(T)$ can be constructed by substituting

$\hat{\lambda}_{ml}(T)$ and X_N into the relationship given in (8):

$$\widehat{Var}\left(\hat{\lambda}_{ml}(T)\,L\right) = \hat{\lambda}_{ml}(T)^2 L^2 / x = xL^2/T^2 . \qquad (9)$$

If n were observable, then $\hat{\lambda}_{ml}(T)\,L$ would equal n and have variance λL. In the present circumstance, however,

$$Var\left(\hat{\lambda}_{ml}(T)\,L\right) = Var\left(E\left(\hat{n}_{ml}(T) \mid N = n\right)\right) + E\left(Var\left(\hat{n}_{ml}(T) \mid N = n\right)\right).$$

The term $E\left(\hat{n}_{ml}(T) \mid N = n\right)$ is approximately n, and the variance of N, a Poisson random variable, is λL. Hence, the variance of $\hat{\lambda}_{ml}(T)\,L$ can be approximated as:

$$Var\left(\hat{\lambda}_{ml}(T)\,L\right) \cong \lambda L + E\left(Var\left(\hat{n}_{ml}(T) \mid N = n\right)\right).$$

Consequently, the estimated variance of $\hat{\lambda}_{ml}(T)\,L$ is:

$$\widehat{Var}\left(\hat{\lambda}_{ml}(T)\,L\right) = \hat{\lambda}_{ml}(T)\,L + \widehat{Var}\left(\hat{n}_{ml}(T) \mid N = n\right).$$

Notice that combining (9) with the result above yields

$$\widehat{Var}\left(\hat{n}_{ml}(T) \mid N = n\right) = x\,L^2/T^2 - xL/T = x\,L\,(L-T)/T^2 ,$$

which is the same as the estimator derived in Section 2.1 using the second difference method.

2.5 Interval Estimation for λ

Confidence limits for λ can be constructed using the unconditional distribution of idle time. A 1-α upper confidence limit for λ is found by solving for λ in the equation:

$$P\left(T_N \geq t; \lambda\right) = 1 - \int_0^t \lambda^{x+1} \exp(-\lambda u)\, u^x \, du \,/\, \Gamma(x+1) = \alpha .$$

One can write the cdf in terms of the incomplete gamma function and solve for λ. Using the unconditional cdf of idle time, a $1-\alpha$ lower confidence limit for λ, is found by solving for λ in the equation

$$P\left(T_N \leq t; \lambda\right) = \int_0^t \lambda^{x^*} \exp(-\lambda u)\, u^{x^*-1} \, du \,/\, \Gamma(x^*) = \alpha ,$$

where $0 \leq t < L$ and $x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x-1} < L\text{-}t < W_x . \end{cases}$

As for confidence limits of n, one can adopt the convention of allowing x^* to equal x+1 for both upper and lower confidence limits of λ. Alternatively, one can use the Poisson form of the cdf or upper tail probability to solve for the lower and upper confidence limits of λ. It should be noted that the $1-2\alpha$ confidence limits on λ derived here produce an open confidence interval, $\lambda_{lower} < \lambda < \lambda_{upper}$, with coverage probability of at least $1-2\alpha$.

3. ESTIMATION WHEN ONLY THE NUMBER OF SERVICES (X) IS OBSERVED

When the number of services is the sole observation and the shift length, L, and the cumulative service time through the x^{th} service, W_x, are known constants, the distribution of X_n, conditional on the number of arrivals, can be used to derive point and interval estimators of n. A maximum likelihood estimator can be derived for the case of unequal service times. For the equal service time case, an approximate method of moments estimator (MOM) has been derived and an estimator constructed to be unbiased for $n < L/w$ can be constructed. Confidence limits for n are based on the conditional distribution of X_n and are derived similarly to those based on the conditional distribution of T_n.

It is possible to use the unconditional distribution of the number of services to derive point and interval estimators of λ, the unknown rate parameter of the Poisson arrival process. Less emphasis has been placed on this, however; if balking is an unobservable feature, it seems more useful to estimate the number of arrivals that occurred rather than the arrival rate. Empirical results confirm that the $1-2\alpha$ confidence intervals for λ are longer than the corresponding intervals for n.

3.1 Maximum Likelihood Estimator of n

The MLE of n, $\hat{n}_{ml}(X)$, does not have a closed form and is most easily found by calculating

$$P(X_n = x; n, L) =$$

$$\sum_{r=0}^{n-x} \binom{n}{x} W_{x-1}^{r} (L-W_{x-1})^{n-r} / L^n - \sum_{r=0}^{n-x-1} \binom{n}{x+1} W_x^r (L-W_x)^{n-r} / L^n$$

for successive values of n until the probability decreases. The value of n, for which $P(X_n = x; n, L)$ is largest, is $\hat{n}_{ml}(X)$.

The finiteness of $\hat{n}_{ml}(X)$ is guaranteed because it is bounded above and below by the largest and smallest values, respectively, of

$$\hat{n}_{ml}(T) = \hat{n}_{ml}(X, Y) = XL/(L-W_x+Y)$$

as a function of Y_n ; thus, the following theorem proves that $P(X_n = x; n, L)$ is maximum for n in the given interval. Using the fact that $f_n(x, y)/f_{n-1}(x, y)$ is a decreasing function of n, which passes through unity at $n = XL/(L-W_x+Y)$ for $0 \le y < w_x$, we can prove by contradiction that the n-solution to $f_n(x) = f_{n-1}(x)$ belongs to the interval given in the theorem below.

Theorem 2. The integer-valued MLE of n, $\hat{n}_{ml}(X)$, satisfies

$$XL/(L-W_{x-1}) \le \hat{n}_{ml}(X) \le XL/(L-W_x)$$

for $W_x < L$ and $0 \le y < w_x$.

3.2 Method of Moments Estimator of n for Equal Service Times

Rubin and Robson (1989a) show that the conditional mean number of services is well approximated by

$$E(X_n) \cong n/(1+(n-1)(w/L)),$$

for the case of equal service times; in fact, the approximation is exact for w=0, L or L/(n-1). A point estimator of n based on this approximation can be derived using the method of moments technique:

$$\tilde{n}(X) = X(L-w)/(L-wX),$$

for $0 < w < L$ and $wX < L$. Notice that $\tilde{n}(X)$ is not necessarily integer-valued. It can be shown algebraically that the MOM estimator conforms to the bounds on $\hat{n}_{ml}(X)$ that were given in Theorem 2 (Rubin, 1987).

Table 2 can be used to compare $\hat{n}_{ml}(X)$ and $\tilde{n}(X)$ and their bounds for several values of w.

3.3 Construction of a Restricted Unbiased Estimator of n

When service times are equal, a unique unbiased estimator of n, $\hat{n}_u(X)$, can be constructed for n over the restricted range $0 \leq n < L/w$. The ability to construct such an estimator capitalizes on the recursive nature of the formula for expectations using the conditional density function of X_n when $n \leq L/w$. The existence and uniqueness of $\hat{n}_u(X)$ are proved in the following theorem. The form of the estimator is given as a corollary to the theorem. To simplify notation, the shift length has been set to unity ($L \equiv 1$).

<u>Theorem 3</u>. When all service times are equal to w and L=1, there exists a unique function of X, say $\hat{n}_u(X)$, which is unbiased for n over the restricted range $0 \leq n < 1/w$.

<u>Proof</u>. We require $\hat{n}_u(X)$ to satisfy

$$E\left(\hat{n}_u(X)\right) \equiv E\left(\hat{n}_u(X) \mid N = n\right) = n;$$

i.e.,

$$n = \sum_{x=0}^{n} \hat{n}_u(x) P\left(X_N = x \mid N = n\right) \text{ for } n = 0, 1, \ldots, [1/w]. \tag{10}$$

At N = 0 we have $P(X_0 = 0 \mid N = 0) = 1$, so (10) becomes

$$0 = \hat{n}_u(0) \times 1 \Rightarrow \hat{n}_u(0) = 0\,.$$

At N = 1 we have $P(X_N \leq 1 \mid N = 1) = 1$, so (10) becomes:

$$1 = \sum_{x=0}^{1} \hat{n}_u(x) P\left(X_N = x \mid N = 1\right)$$

$$= \binom{1}{0} 0^0\, 1^1 \left\{\hat{n}_u(1) - \hat{n}_u(0)\right\} = \hat{n}_u(1).$$

$$\Rightarrow \hat{n}_u(1) = 1\,.$$

At $N = 2$ we have $P(X_N \leq 2 \mid N = 2) = 1$, so (10) becomes:

$$2 = \sum_{x=0}^{2} \hat{n}_u(x)\, P\,(X_N = x \mid N = 2)$$

$$= 1 + (1\text{-}w^2)\left\{\hat{n}_u(2) - 1\right\}.$$

$$\Rightarrow \hat{n}_u(2) = 1 + 1/(1\text{-}w^2) = \sum_{r=0}^{1} 1/(1-rw)^2.$$

At $N = k$ we have $P(X_N \leq k \mid N = k) = 1$, so recursively solving (10) gives:

$$\hat{n}_u(k) = k - \left\{\sum_{x=0}^{k-1} \hat{n}_u(x)\, P\,(X_N = x \mid N = k)\right\} / P\,(X_N = k \mid N = k), \tag{11}$$

where, for $0 < x \leq k \leq 1/w$, $P(X_N = k|N = k) > 0$, ensuring the finiteness of (11). Therefore, a unique unbiased estimator of n exists for $0 \leq n < 1/w$.

Alternatively, one can prove Theorem 3 using a completeness argument. The form of $\hat{n}_u(X)$ is derived as a corollary to Theorem 3; the proof is an induction argument that uses the identity:

$$\sum_{x=0}^{n} \binom{n}{x} (xw)^{n-x} (1\text{-}xw)^{x-1} = 1 \qquad \text{for } 0 < w < 1/n.$$

For details, see Rubin (1987).

Corollary 3.1. When all service times are equal to w and L=1, the unique unbiased estimator of n, for $0 \leq n < 1/w$, is

$$\hat{n}_u(X) = \sum_{r=0}^{X-1} 1/(1-rw)^2,$$

where $0 < w < 1$ and $x \geq 1$.

Numerical results indicate that a well-behaved, closed-form approximation to the unbiased estimator is found by integration, i.e.,

$$\hat{n}_u(X) = \sum_{r=0}^{X-1} 1/(1-rw)^2$$

$$\cong \int_{-0.5}^{X-0.5} (1-rw)^{-2}\,dr = X/\{(1+0.5\,w)(1-(X-0.5)\,w)\}.$$

The approach taken in Corollary 3.1 can be used to produce an estimator of the $\mathrm{Var}(\hat{n}_u(X))$ which is unbiased for $n < 1/w$. An unbiased estimator of $C_2^n = \binom{n}{2}$ will be constructed; from this we get an unbiased estimator of n^2. It can be shown that the estimated variance of $\hat{n}_u(X)$ is:

$$\mathrm{Var}(\hat{n}_u(X)) = (\hat{n}_u(X))^2 - 2\,\hat{C}_2^n(X) - \hat{n}_u(X),$$

where $\hat{C}_2^n(X)$ is an unbiased estimator of $\binom{n}{2}$ for $n < 1/w$.

The proof of the existence of a unique unbiased estimator of $\binom{n}{2}$ for $n<1/w$ and the construction of that estimator follow the pattern established for $\hat{n}_u(X)$. The form of $\hat{C}_2^n(X)$ is given by an induction argument, requiring the identity:

$$n\binom{N}{n}\sum_{r=0}^{N-n}\binom{N-n}{r}(-1)^r(r+n+1)^{N-n}/(r+n) = 1 \qquad \text{for } n = 1, 2, \ldots, N,$$

which follows from Rubin (1987).

Corollary 3.2. When all service times are equal to w and L=1, the unique unbiased estimator of $\binom{n}{2}$, for n = 2, . . . , [1/w], is

$$\hat{C}_2^n(X) = \sum_{r=1}^{X-1} r(1-w)/(1-rw)^3,$$

where 0 <w <1 and x ≥ 2.

Numerical results indicate that a well-behaved, closed-form approximation to $\hat{C}_2^n(X)$ is found by integration:

$$\hat{C}_2^n(X) \cong \int_{0.5}^{X-0.5} \frac{r(1-w)}{(1-rw)^3}\, dr = \left(\frac{1-w}{2w^2}\right)\left\{\frac{2w(X-0.5)-1}{(1-(X-0.5)w)^2} + \frac{1-w}{(1-0.5w)^2}\right\}.$$

Since $E(\hat{C}_2^n(X)) = n(n-1)/2$ and $E(\hat{n}_u(X)) = n$,

$$n^2 = 2E\left(\hat{C}_2^n(X)\right) + E\left(\hat{n}_u(X)\right).$$

Thus,

$$\widehat{n^2} = 2\hat{C}_2^n(X) + \hat{n}_u(X)$$

and

$$\mathrm{Var}\left(\hat{n}_u(X)\right) = \hat{n}_u(X)\left\{\hat{n}_u(X) - 1\right\} - 2\hat{C}_2^n(X).$$

The estimated variance of $\hat{n}_u(X)$ is guaranteed to be nonnegative for all values of X. Plotting $\hat{n}_u(X)$ versus $P(X_n \geq x) = P(\hat{n}_u(X) \geq \hat{n}_u(x))$ yields a virtually straight line, indicating that $\hat{n}_u(X)$ is nearly lognormal.

3.4 Interval Estimation for n

The upper tail probability of the number of services, conditional on the number of arrivals, given by

$$P(X_n \geq x; L) = x\binom{n}{x}\int_0^{1-(W_{x-1}/L)} u^{x-1}(1-u)^{n-x}\,du \qquad \text{for } W_{x-1} < L,$$

can be used to construct confidence limits for n. Proceeding as in the case of T_n, we transform the incomplete beta probability with parameters x+1 and n-x to an F probability with 2(x+1) and 2(n-x) degrees of freedom. Solving the equation

$$\frac{n-x}{x+1}\left(\frac{L-W_x}{W_x}\right) = F_{2(x+1),\,2(n-x)}(1-\alpha)\,. \tag{12}$$

for n yields the 1-α upper confidence limit for n. Notice that (12) is equal to (5), the equation that yields the 1-α upper confidence limit for n based on T. For a 1-α lower confidence limit on n, we solve for n using the equation:

$$\frac{n-x+1}{x}\left(\frac{L-W_{x-1}}{W_{x-1}}\right) = F_{2x,\,2(n-x+1)}(\alpha)\,. \tag{13}$$

Integer-valued solutions to (12) and (13) can be determined using F-tables. An exact solution, which is noninteger, can be computed using the F cdf and a root finding algoritm. Since $x \leq n$, if $n_{lower} < x$, we replace n_{lower} with x. This procedure provides a 1-2α confidence interval which is open $(\max(x, n_{lower}) < n < n_{upper})$ and has coverage probability of at least 1-2α.

The 1-2α confidence interval for n based on X is longer than the corresponding interval based on T for all $\alpha \in (0, 1)$. The 1-α upper confidence limit for n based on X and that based on T are equal. The 1-α lower confidence limits for n based on X and on T use different beta density functions as their kernals. The ratio of these functions, given by

$$\frac{g_X(u)}{g_T(u)} = \frac{x\binom{n}{x}u^{x-1}(1-u)^{n-x}}{(n-x)\binom{n}{x}u^{x}(1-u)^{n-x-1}} = \left(\frac{x}{n-x}\right)\left(\frac{1-u}{u}\right),$$

is a strictly decreasing function of u. The monotonicity of the ratio of the kernals and the fact that

$$\int_0^1 g_X(u)\,du = \int_0^1 g_T(u)\,du = 1$$

imply that

$$c_X < c_T \qquad \text{for all } \alpha \in (0, 1),$$

where $\alpha = P(X_n \le c_X) = P(T_n \le c_T)$. Thus, the 1-$\alpha$ lower confidence limit for n based on X is smaller than the corresponding lower limit based on T for all α. Consequently, the confidence interval for n based on X is longer than that based on T.

3.5 Maximum Likelihood Estimator of λ

The unconditional distribution of X_N can be used to derive the MLE of λ, $\hat{\lambda}_{ml}(X)$. Notice that, for x > 0, the unconditional density function is the difference between two gamma cdfs with the same shape parameter but different location parameters:

$$\int_0^{L-W_{x-1}} \frac{\lambda^x}{\Gamma(x)} u^{x-1} \exp(-\lambda u)\,du - \int_0^{L-W_x} \frac{\lambda^{x+1}}{\Gamma(x+1)} u^{x} \exp(-\lambda u)\,du,$$

where $\lambda>0$ and $W_{x-1} \le W_x < L$. For x=0 and $\lambda>0$,

$$P\left(X_N = 0; \lambda, L\right) = \exp\left(-\lambda\right),$$

since $P(X_N \geq 0) \equiv 1$ and $P(X_N \geq 1) = 1 - e^{-\lambda}$.

Setting the derivative of the density function with respect to λ equal to zero yields:

$$\frac{(L-W_x+w_x)^x}{\Gamma(x)} \exp\left(-\lambda(L-W_x+w_x)\right) - \frac{(L-W_x)^{x+1}}{\Gamma(x+1)} \lambda^x \exp\left(-\lambda(L-W_x)\right) = 0 .$$

The MLE of λ is the solution to the equation:

$$\ln(x/(L-W_x)) + x\ln(1+w_x/(L-W_x)) - \ln(\lambda) - \lambda w_x = 0 .$$

Evaluating the second derivative of the likelihood function with respect to λ at the point for which the first derivative is equal to zero yields:

$$-\frac{(L-W_x)^{x+1}}{\Gamma(x+1)} \lambda^x \exp\left(-\lambda(L-W_x)\right)\left(w_x + 1/\lambda\right) < 0 ,$$

which implies that the likelihood is maximum at this point (and the maximum is unique.)

If one wishes to exclude the outcome N=0, one can use a truncated Poisson distribution (N≥1) as the basis for deriving a maximum likelihood estimator of λ (Rubin, 1987).The MLE based on the truncated distribution is smaller than $\hat{\lambda}_{ml}(X)$ for all values of X>0. Rubin (1987) also gives confidence limits of λ based on the truncated Poisson distribution.

Table 2 illustrates $\hat{n}_{ml}(X)$ and $\hat{\lambda}_{ml}(X)$ along with upper and lower confidence limits of n and λ, for several values of x with w = 0.1 and L=1.

3.6 Interval Estimation for λ

The unconditional upper tail probability of X_N can be used to derive upper and lower confidence bounds for λ. Setting the upper tail probability, written in terms of the incomplete gamma function, equal to α, allows us to solve for the 1-α lower confidence limit of λ:

$$\alpha = \int_0^{\lambda(L-W_{x-1})} z^{x-1} \exp(-z)\, dz \,/\, \Gamma(x).$$

Likewise, the solution to the equation

$$1-\alpha = \int_0^{\lambda(L-W_x)} z^{x} \exp(-z)\, dz \,/\, \Gamma(x+1)$$

yields a 1-α upper confidence limit for λ. Notice that this equation is the same as that used to solve for the 1-α upper confidence limit for λ based on T (see Section 2.5).

Table 2 illustrates the 1-α upper and lower confidence limits for λ, as well as the corresponding limits of n, for several values of x with w =0.1 and L=1. As one expects, the confidence intervals for λ are longer than the corresponding intervals for n, since the unconditional distribution incorporates more variability in X than does the conditional distribution.

The 1-2α confidence interval for λ based on X is longer than the corresponding interval based on T for all $\alpha \in (0, 1)$. This can be shown with an argument, similar to that for confidence intervals for n based on X and on T, which exploits the monotonicity of the ratio of the gamma kernals for X and T.

4. ESTIMATION WHEN THE NUMBER OF BALKERS (Z=n-X) IS OBSERVED

Estimation of the number of arrivals or the arrival rate is not possible for the case of unequal service time, when only the number of balkers (Z=n-X) is observed, since the cumulative service time through the x^{th} service, W_x, is unknown.

If the service times are equal for all customers and the shift length and common service time are known, one can estimate the number of arrivals, or equivalently, the number of services, when only the number of balkers is observed. Notice that, conditional on the number of arrivals, Z_n is a simple transformation of X_n, and $P(Z_n \leq z) = P(X_n \geq n\text{-}z)$. Equating the cdf of Z_n to α for a fixed z yields a 1-α upper confidence limit for n. As in the case for X_n, transforming the beta probability to an F probability gives an equation in terms of a cutoff point and a critical value of an F distribution, which must be solved iteratively for n to yield the 1-α upper confidence limit for n:

$$\frac{n-z}{z+1}\left(\frac{(n-z-1)\,w}{L-(n-z-1)\,w}\right) = F_{2(z+1),\,2(n-z)}(1-\alpha)\,. \tag{14}$$

The inference target in this circumstance might be X rather than n, but since X = n-Z and Z is observed, any inference about n carries with it an inference about X. Thus, (14) can be written as:

$$\frac{x}{z+1}\left(\frac{(x-1)\,w}{L-(x-1)\,w}\right) = F_{2(z+1),\,2(x)}(1-\alpha)\,. \tag{15}$$

A 1-α lower confidence limit of n is found by solving the following equation for n:

$$\frac{z}{(n-z+1)}\left(\frac{L-(n-z)\,w}{(n-z)\,w}\right) = F_{2(n-z+1),\,2z}(\alpha)\,. \tag{16}$$

Integer-valued solutions to (14) and (16) can be determined from F-tables. An exact solution, which is noninteger, can be computed using the F cdf and a root finding algorithm.

Setting $\alpha = 0.5$ and solving for n in equation (14) or x in (15) yield median unbiased point estimators of n and X, respectively (Lehmann, 1983). Figure 1 shows the median unbiased estimators of X flanked by their corresponding 90% and 95% upper and lower confidence limits for w=0.01 and L=1.

An attempt was made to construct an unbiased estimator of n using the conditional distribution of Z_n. The procedure used was similar to that detailed in Section 3.3. Unfortunately, the estimator is badly-behaved. The unbiased estimator, $\hat{n}_u(Z)$, fluctuates wildly and even takes on values that are outside of the range of n. For example, with w=0.1, L=1, n=3 and Z=2, $\hat{n}_u(Z) = -92.23$. Therefore, $\hat{n}_u(Z)$ is unacceptable as an estimator and will not be considered further.

Estimation of λ, the rate parameter of the Poisson arrival process, is intractable when only the number of balkers is observed. For instance, the distribution of Z_N, conditional on X_N, is given by:

$$P(Z_N=z|X_N=x) = \exp(-\lambda L)\{(\lambda L)^z/z!\}\left\{\int_{1-xw/L}^{1-(x-1)w/L} u^{x-1}(1-u)^z du\right.$$

$$\left.+\int_0^{1-xw/L} u^{x-1}(1-u)^{z-1}(1-xu-zu)\,du\,/x\right\}\Big/\int_{1-xw/L}^{1-(x-1)w/L} u^{x-1}\exp(-\lambda Lu)\,du,$$

for $x > 0$, $\lambda > 0$ and $0 < w < L$. In addition to being complicated, this distribution depends on both x and λ. Notice that if both x and z are observed, then n is known and $\hat{\lambda}_{ml} = n$.

5. ESTIMATION OF SHIFT LENGTH (L)

Estimation of shift length, L, is possible for the case of unequal service times when the data observed are (n, X), (n, X, Y), (n, T) or (n, S). Point and interval estimators are derived using distributional results that are conditional on the number of arrivals.

Recall that when the sequence of service times through the x^{th} service is known, the distributions of (X, Y), T and S are equivalent. For L and W_x known, T and S are still equivalent since $S = W_x - Y$ is a known univariate transformation of T that is one to one. It is not possible, however, to resurrect (X, Y) from T when only W_x is known.

5.1 Point Estimators of L Based on (n, T), (n, S) or (n, X, Y)

Differentiating the conditional likelihood of total server idle time, T, or server busy time, S, with respect to L yields:

$$\hat{L}_{ml}(S, n) = n\,S/(n - x(S))$$

or

$$\hat{L}_{ml}(T, n) = n\,T/x$$

for $0 < S = L\text{-}T < W_n$ and $n \geq 1$ with

$$x(S) = \begin{cases} x & \text{if } S = W_x \\ x\text{-}1 & \text{if } W_{x\text{-}1} < S < W_x. \end{cases}$$

From the joint likelihood of X_n and Y_n, conditional on n arrivals ($n > 0$), we find

$$\hat{L}_{ml}(X, Y, n) = \begin{cases} W_x/(1- X/n) & \text{for } Y = 0 \\ (W_x\text{-}\ Y)/(1\text{-}\ (X\text{-}1)/n) & \text{for } 0 < Y < w_x \end{cases}$$

for $X > 0$ and $n > 0$.

5.2 Confidence Limits for L Based on (n, T) or (n, S)

The construction of 1-α confidence limits for L based on (n, T) or (n, S) is similar to that done for n based on observation of T when the sequence of service times through the x^{th} service is known. Equations (4) and (5) of Section 2.3 can be used to calculate upper and lower confidence limits of L, when (L-S)/L is substituted for T/(L-T) and x^*=x(S)+1. Consequently, confidence limits of L are obtained by solving for L in the following equation:

$$F_{2x^*,\,2(n\text{-}x^*+1)}(\alpha) = (n\text{-}x^*+1)\,t/x^*(L\text{-}t) = \{(L\text{-}S)/\hat{L}_{ml}(S, n)\}/\{x^*/n\}$$

$$\text{for } x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x\text{-}1} < L\text{-}t < W_x. \end{cases}$$

Thus, the 1-α upper confidence limit of L is given by:

$$L_{upper}(S) = S + (x^* / n)\,\hat{L}_{ml}(S, n)\; F_{2x^*,\, 2(n-x^*+1)}(1-\alpha).$$

Notice that the upper confidence limit is the sum of the observed busy time, S, and the estimated idle time in a workshift of unknown duration L, $(x^* / n)\,\hat{L}_{ml}(S, n)$. Note that $\hat{L}_{ml}(S, n)$ and $L_{upper}(S)$ are infinite for $S = W_n$, when the number of services equals the number of arrivals. The 1-α lower confidence limit of L is given by:

$$L_{lower}(S) = S + (S\, x(S) / (n-x(S)))\; F_{2x(S),\, 2(n-x(S)+1)}(\alpha),$$

where x(S) is the number of services during the busy period (= x^*-1). $L_{lower}(S)$ is always finite.

6. SUMMARY

Estimation based on observation of a single shift from the queueing process has been considered for several situations in which incomplete data are collected. When total server idle time is observed, point and interval estimators of the number of arrivals (n) and the arrival rate (λ) are derived. When only the number of services is observed, interval estimators and a variety of point estimators of n and λ are derived. For the case of equal service time, point estimators of n, based on T or X, can be constructed to be unbiased over the restricted range of $n < L/w$. When the number of balkers is observed, estimation of n or λ is possible only for the equal service time case; a median unbiased point estimator and interval estimators of n have been derived.

In addition, when n and X or n and T are observed, the distributions that are conditional on n can be used to estimate shift length (L). Both maximum likelihood estimators and confidence limits can be derived.

REFERENCES

Assefi, T. (1979), Stochastic Processes and Estimation Theory with Applications, New York: John Wiley and Sons.

Basawa, I. V. and Prakasa Rao, B. L. S. (1980), Statistical Inference for Stochastic Processes, London: Academic Press, Inc..

Lehmann, E. L. (1983), Theory of Point Estimation, New York: John Wiley and Sons.

Robson, D. S. and Jones, C. (1989), "The Theoretical Basis of an Access Site Angler Survey Design," Biometrics 45, 83-98.

Rubin, G. (1987), "Statistical Distribution and Estimation Theory for a Single Server Queue with Fixed Service Time and Complete Balking," unpublished Ph.D. Dissertation, Cornell University, Ithaca, NY.

Rubin, G. and Robson, D.S. (1989a), "Estimation Theory for a Single Server Queue with Random Arrivals and Complete Balking," Technical Report #BU-1019-M in Biometric Series, Cornell University.

______(1989b), "Estimation of Service Times for Single Server Queues With Balking," Technical Report #BU-1026-M in Biometric Series, Cornell University.

Samaan, J. E. and Tracy, D.S. (1979), "On the conditional estimation for a parameter of a queueing system with loss," Computer Science and Statistics: 12th Annual Symposium on the Interface, ed. J. F. Gentleman, Waterloo: University of Waterloo, pp. 324-327.

LIST OF TABLES

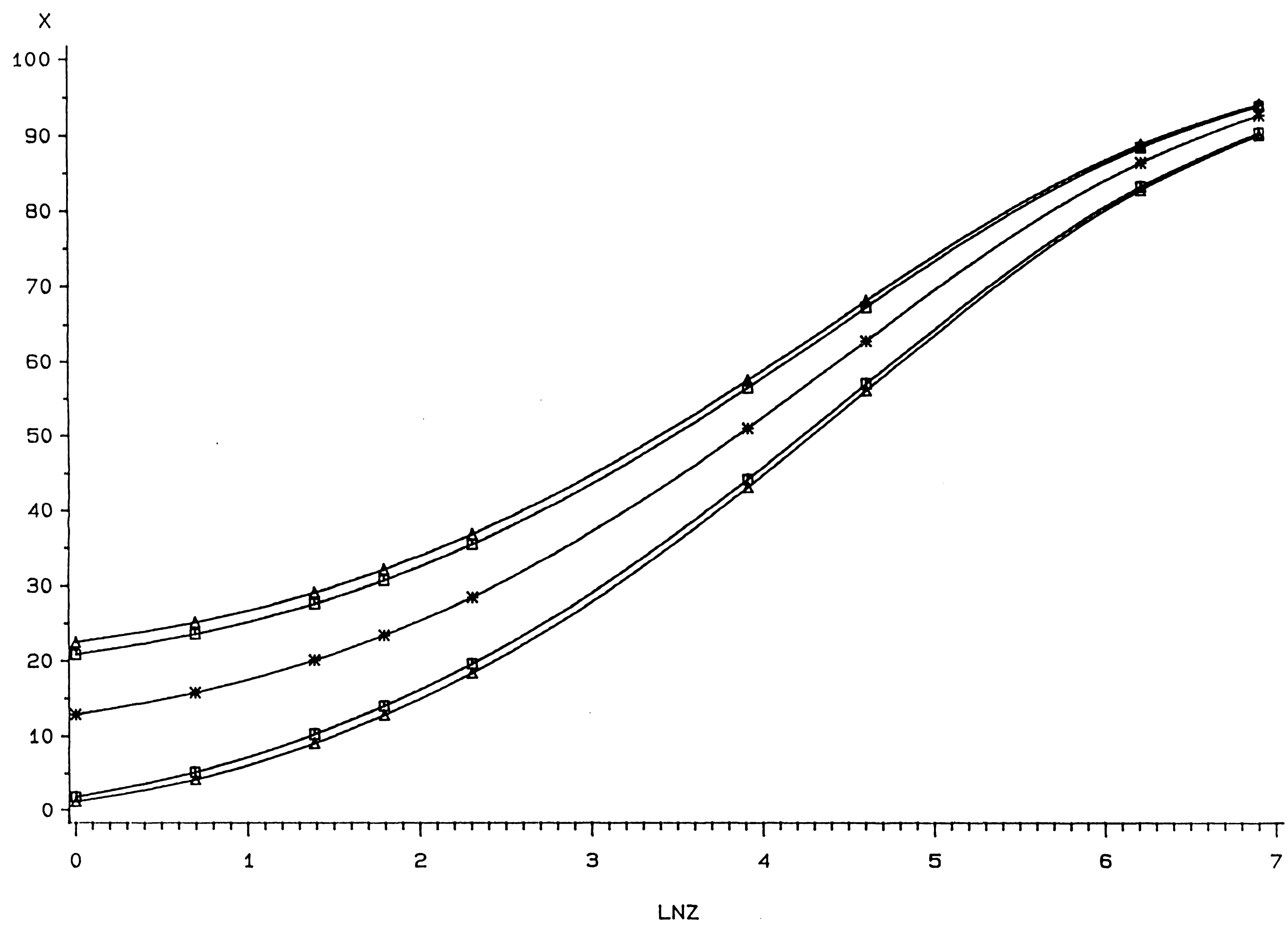
Table 1. Two point estimators of n, $\hat{n}_{ml}(X)$ and $\tilde{n}(X)$, and their lower and upper bounds, $\hat{n}_{ml}(X, Y = w^{-})$ and $\hat{n}_{ml}(X, Y = 0)$, respectively, calculated for several values of w and X with L = 1.

w	X	$\hat{n}_{ml}(X, Y = w^{-})$	$\hat{n}_{ml}(X)$	$\tilde{n}(X)$	$\hat{n}_{ml}(X, Y = 0)$
0.60	1	1.0000	2	1.0000	2.5000
0.40	1	1.0000	1	1.0000	1.6667
0.40	2	3.3333	6	6.0000	10.0000
0.20	1	1.0000	1	1.0000	1.2500
0.20	2	2.5000	3	2.6667	3.3333
0.20	3	5.0000	6	6.0000	7.5000
0.20	4	10.0000	19	16.0000	20.0000
0.20	5	25.0000	∞	∞	∞
0.10	1	1.0000	1	1.0000	1.0000
0.10	2	2.2222	2	2.2500	2.5000
0.10	3	3.7500	4	3.8571	4.2857
0.10	4	5.7143	6	6.0000	6.6667
0.10	5	8.3333	9	9.0000	10.0000
0.10	6	12.0000	13	13.5000	15.0000
0.10	7	17.5000	21	21.0000	23.3333
0.10	8	26.6667	33	36.0000	40.0000
0.10	9	45.0000	65	81.0000	90.0000
0.10	10	100.0000	∞	∞	∞

Table 2. Maximum likelihood estimators of n and λ, $\hat{n}_{ml}(X)$ and $\hat{\lambda}_{ml}(X)$, and the corresponding 95 % upper and lower confidence limits for n and λ, when w = 0.1 and L = 1.

X	$\lambda_L(X)$	$n_L(X)$	$\hat{\lambda}_{ml}(X)$	$\hat{n}_{ml}(X)$	$\lambda_U(X)$	$n_U(X)$
1	0.05	1	1.11	1	5.271	2.71
2	0.40	2	2.47	2	7.870	5.12
3	1.02	3	4.20	4	11.077	8.18
4	1.95	4	6.47	6	15.256	12.24
5	3.28	5	9.56	9	21.026	17.91
6	5.27	6.79	14.05	13	29.606	26.40
7	8.21	9.86	21.13	21	43.827	40.54
8	13.27	14.99	34.05	33	72.173	68.82
9	23.48	25.27	65.55	65	157.052	153.60
10	54.25	56.11	∞	∞	∞	∞

LIST OF FIGURES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